MMP Learning Seminar.

Week 56:

Ascending chain condition for log canonical thresholds

Ascending chain condition for log canonical thresholds:

(X,Δ) a log canonical pair, $M \geq 0$ $\mathbb{R}$-Cartier on X

The log canonical threshold of M wrt to (X,Δ) is

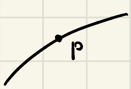
$$\mathrm{lct}\,(X,\Delta; M) = \sup\{ t \in \mathbb{R} \mid (X, \Delta + tM) \text{ is lc}\}$$

$\mathcal{I}_n(I) =: \mathcal{I}$ is the set of (X,Δ) where X has dim n,

(X,Δ) is lc, $\mathrm{coeff}(\Delta) \subseteq I$.

$$LCT_n(I,J) = \{ \mathrm{lct}(X,\Delta; M) \mid (X,\Delta) \in \mathcal{I}_n(I) \}$$

& the coefficients of M are in J.



$\mathbb{A}^1$ $\quad c\{0\}$ $\qquad \mathrm{lct} = (\mathbb{A}^1; c\{0\}) = \frac{1}{c}$.

Theorem 1.1: Fix $n \in \mathbb{N}$, $I \subseteq [0,1]$ and $J \subseteq \mathbb{R}_{\geq 0}$.

If I & J are DCC, then $LCT_n(I,J)$ satisfies the ACC.

Corollary 1.2: Assume termination of flips for $\mathbb{Q}$-factorial klt pairs in dimension $n-1$.

Let (X,Δ) klt pair with X $\mathbb{Q}$-factorial projective of dim n.

If $K_X+\Delta \equiv D \geq 0$, then any sequence of $(K_X+\Delta)$-flips terminates.

$$\left.\begin{array}{c}\text{MMP in dim } n-1 \\ + \\ \text{ACC for lct's} \\ \text{in dim } n\end{array}\right\} \Longrightarrow \text{MMP for effective pairs in dim } n.$$

Theorem 1.3: $\{(X,\Delta) \text{ lc} \mid X \text{ of dim } n, \ \operatorname{coeff}(\Delta) \subseteq I \text{ DCC}\} =: \mathfrak{D}$.

There exists $\delta \geq 0$ and m an integer s.t.:

(1) the set $\{\operatorname{vol}(X, K_X+\Delta) \mid (X,\Delta) \in \mathfrak{D}\}$ satisfies the DCC.

Further, if $(X,\Delta) \in \mathfrak{D}$ and $K_X+\Delta$ is big.

(2) $\operatorname{vol}(X, K_X+\Delta) > \delta$.

(3) $\phi_{m(K_X+\Delta)}$ is birational

"To bound general type varieties, we need to control their volume".

$$\phi_{m(K_X+\Delta)} := \phi_{\lfloor m(K_X+\Delta)\rfloor}$$

Theorem 1.5 (Global ACC): Fix $n \in \mathbb{Z}_{>0}$, $I \subseteq [0,1]$ which satisfies the DCC.

There exists $I_0 \subseteq I$ finite with the following conditions:

If (X, Δ) is lc such that:

(1) X is projective of dimension n,

(2) (X, Δ) is lc,

(3) $\mathrm{Coeff}(\Delta) \subseteq I$,

(4) $K_X + \Delta \equiv 0$.

Then, $\mathrm{coeff}(\Delta) \subseteq I_0$.

Exercise: prove this statement in $\mathbb{P}^1$!

Corollary 1.7 (Boundedness of CY FT varieties):

Fix $n \in \mathbb{Z}_{>0}$, $\varepsilon > 0$, and I a DCC set.

Let $\mathcal{D}$ be the set of all pairs (X, Δ), where:

- X is projective of dim n,
- $\mathrm{coeff}(\Delta) \subseteq I$
- the log discrepancies of (X, Δ) are $> \varepsilon$.
- $K_X + \Delta \equiv 0$, and
- $-K_X$ is ample.

"To bound Fano varieties, we need to control the singularities".

Then $\mathcal{D}$ forms a bounded family.

Fano index: (X, Δ) a lc pair, X proj of dim n and $-(K_X+\Delta)$ is ample. The Fano index of (X,Δ) is the largest real number r such that

$$-(K_X+\Delta) \sim_{\mathbb{R}} rH \qquad K_X+\Delta+rH \sim_{\mathbb{Q}} 0.$$

is still nef.

where H is a Cartier divisor.

By Kobashi-Ochiai the Fano index is at most $\dim(X)+1$.

Warning: The definition is not well-behaved if replace Cartier with Weil.

$R_n(I)$ = the set of all Fano indices of dim n with $\operatorname{coeff}(\Delta) \subseteq I$.

The Fano spectrum of I in dimension n.

Corollary 1.10: $I \subseteq [0,1]$ satisfies the DCC & $n \in \mathbb{Z}_{>0}$, then $R_n(I)$ satisfies the ACC.

Theorem 1.11: If 1 is the only accumulation point of $I \subseteq [0,1]$ and $I = I^+$ then the accumulation points of $LCT_n(I)$ are $LCT_{n-1}(I) - \{1\}$. In particular, if $I \subseteq \mathbb{Q}$, then the accumulation points of $LCT_n(I)$ are in $\mathbb{Q}$.

The Main Theorems:

Theorem A: ACC for lct's.

+1

Theorem B: Upper bound for volumes.

$K_X+\Delta \equiv 0$. $(X,\Delta)\in\mathfrak{D}_{klt}$, then $\mathrm{vol}(\Delta) < v_0$.

Theorem C: Birational boundedness.

Theorem D: Global ACC.

Thm $D_{n-1} \Longrightarrow$ Thm A_n. (5)

Thm D_{n-1} + Thm $A_{n-1} \Longrightarrow$ Thm B_n (6)

Thm C_{n-1} + Thm A_{n-1} + Thm $B_n \Longrightarrow$ Thm C_n (7)

Thm D_{n-1} + Thm $C_n \Longrightarrow$ Thm D_n (8).

Example: $X_{p,q,r} = \mathbb{P}(p,q,r)$.

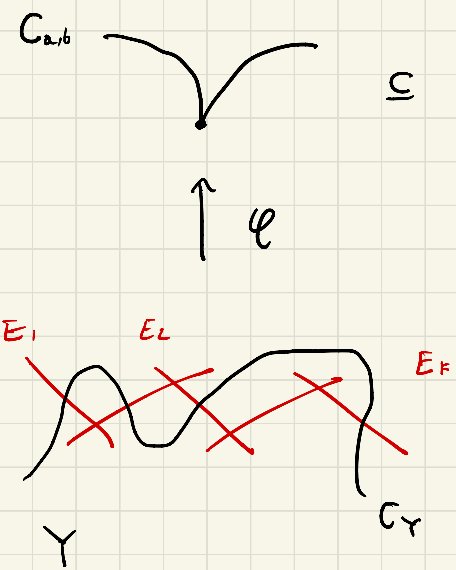
$\Delta :=$ sum of three coordinate lines.

$(X_{p,q,r}, \Delta)$ is lc $K_{X_{p,q,r}}+\Delta \sim 0$. However

$$\mathrm{vol}(\Delta) = \frac{(p+q+r)^2}{pqr}.$$

$\left\{ \frac{(p+q+r)^2}{pqr} \;\middle|\; (p,q,r)\in\mathbb{N}^3 \right\}$ is dense in $\mathbb{R}_{>0}$.

From global ACC to ACC for lct's:



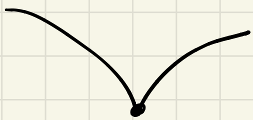
$\subseteq \mathbb{C}^2$

$$C = (y^a + x^b = 0)$$

$$(\mathbb{C}^2, tC)$$

$$\varphi^*(K_{\mathbb{C}^2} + tC) =$$
$$K_Y + f_1(t)E_1 + \dots + f_k(t)E_k + tC_Y$$

increase t until $f_i(t) = 1$ for some i.

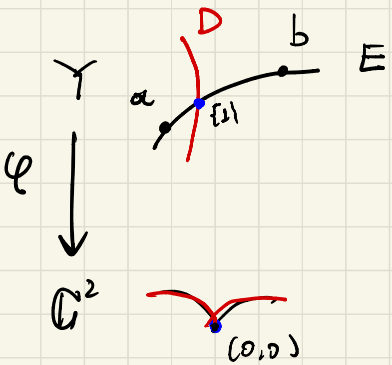


$C = (y^a + x^b = 0) \subseteq \mathbb{C}^2$

Let's say $c \in \mathbb{R}_{>0}$ is the lct.

$(\mathbb{C}^2, cC) \leftarrow$ strictly log canonical

we will extract a unique divisor E over $\mathbb{C}^2$ which is a log canonical place of $(\mathbb{C}^2, cC)$.



Remark: In this case, $Y \to \mathbb{C}^2$ is just the blow-up (a,b).

$$\varphi^*(K_{\mathbb{C}^2} + cC) = K_Y + E + cD \,|_E$$

$$= K_E + \left(\frac{a-1}{a}\right)\{0\} + \left(\frac{b-1}{b}\right)\{\infty\} + c\{1\}$$

$$-2 + \frac{a-1}{a} + \frac{b-1}{b} + c = 0 \implies c = \frac{1}{a} + \frac{1}{b}.$$

lct $(X;\Delta)$

$$(X, c\Delta) \xleftarrow{\text{dlt mod}} (Y, E_1 + \dots + E_r + c\Delta_Y)$$

$$\underset{\substack{\text{|||} \\ 0}}{K_Y + E_1 + \dots + E_r + c\Delta_Y}\big|_{E_i} \equiv K_{E_i} + \underbrace{\Gamma_{E_i}}_{\text{diff}} + c\,\Delta_{E_i}$$

This essentially accounts for $D_{n-1} \Longrightarrow A_n$.

$\square$

"$C_n \Longrightarrow D_n$"

(X, Δ) lc, $\operatorname{coeff}(\Delta) \subseteq I$, $K_X + \Delta \equiv 0$

Run MMP, to reduce to the case $\rho(X) = 1$

so X Fano, Δ is ample "Assume (X, Δ) klt".

increase coeff of $\Delta \leq \Lambda$ now $K_X + \Lambda$ is ample

$|m(K_X + \Lambda)|$ defines a birational map for fixed m.

$K_X + \Lambda_{\lfloor m \rfloor}$ is big.

$\Lambda_{\lfloor m \rfloor} :=$ largest effective divisor $\leq \Lambda$ so that

$m \Lambda_{\lfloor m \rfloor}$ is integral.

This will force $\Delta \leq \Lambda_{\lfloor m \rfloor}$ constraints on the Weil indices of Δ.

$\square$

"$B_m \Longrightarrow C_m$".

$K_X+\Delta$ is big. (X,Δ) lc & $\mathrm{coeff}(\Delta) \subseteq I$ DCC

To apply the Hacon-McKernan strategy:

i) $K_X+\Delta|_V$ we have volume bounded below.

ii) Δ has finite or standard coefficients $1-\frac{1}{m}$.

i) $W \xrightarrow{\text{norm}} V$, $(K_X+\Delta)|_W = K_W + \Theta_b + J$

nice coefficients　　　moduli.

V general enough, $K_W + \Theta_b$ is big, $\mathrm{vol}(K_W+\Theta_b) > \varepsilon$

only depends on dim n

ii) $K_X+\Delta$ big $\Longrightarrow$ $K_X+\Delta_{\lfloor p \rfloor}$ is big. $p := p(I)$.

$$\lambda = \inf\{t \in \mathbb{R} \mid K_X + t\Delta \text{ is big}\}.$$

The problem reduces to control λ away from 1.

$\rho(X)=1$, $K_X+\lambda\Delta \equiv 0$, klt, Thm B $\Longrightarrow$ $\mathrm{vol}(\Delta)<1$.

Using log birational boundedness we aim to control λ. $\square$

"$D_{n-1} + A_{n-1} \Longrightarrow C_n$".

(X,Δ) klt, $K_X + \Delta \equiv 0$, $\mathrm{vol}(\Delta)$ large.

$0 \leq \Pi \sim_{\mathbb{Q}} \lambda\Delta$. λ is small, Π has large mult at a general point.

(X,Π) not klt, Φ close to Δ, (X,Φ) not klt.

$$\begin{array}{ccc} S \subseteq Y & \text{of log discrepancy } 0 \text{ wrt } (X,\Phi). & \\ \downarrow & \left.\begin{array}{c} K_Y + S + \Delta_Y \\ -(K_Y + S + (1-\varepsilon)\Delta_Y) \end{array}\right\} & \text{both ample} \\ (X,\Phi) & & \end{array}$$

ε arbitrarily close to 0.

adjunction to S and turn $(Y, S + (1-\varepsilon)\Delta_Y)$ into a CY pair to obtain a contradiction of the global ACC $\square$.